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A Piecewise Polynomial Lacunary Interpolation Method

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1. INTRODUCTION

This paper is concerned with the following general lacunary interpolation problem:

PROBLEM 1.1. Let

$$\Delta = \{x_1 < x_2 < \cdots < x_{n+1}\}, \tag{1.1}$$

and suppose $I = (v_1, ..., v_p)$ is a vector of integers with $0 \leq v_1 < \cdots < v_p$. Suppose $\{f_i^{v_j}\}_{j=1, i=1}^{p, n+1}$ are given real numbers. Find a function f defined on $[x_1, x_{n+1}]$ such that

$$D^{v_j} f(x_i) = f_i^{v_j}, \qquad j = 1, ..., p \text{ and } i = 1, ..., n+1. \tag{1.2}$$

Special lacunary interpolation problems of this type have been studied by a number of authors over the past 25 years. The subject was of particular interest to the Hungarian group led by Turán (see [1, 13]). Their work concentrated on the use of polynomials to solve the $(0, i)$ interpolation problem, mostly with $i = 2$. In the 1970s and 1980s several papers appeared [2, 8–11, 14–15] in which the $(0, 2)$ problem was solved using polynomial splines.

Recently the first-named author has developed special piecewise polynomial methods for solving $(0, 2)$, $(0, 2, 3)$, and $(0, 2, 4)$ problems; see

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[3–7]. The present paper is an outgrowth of this work. In particular, we show that the approach can be used to define constructive methods for solving the general lacunary interpolation problem, and that for typical smooth classes of functions, the methods deliver optimal-order approximations.

The paper is organized as follows. In section 2 we discuss the basic method in the case $v_1 = 0$. Section 3 is devoted to an error analysis, while in Section 4 the choice of certain required quadrature formulae is discussed. Section 5 deals with the general problem with $0 \leqslant v_1 < \cdots < v_p$. In Section 6 we present examples. Finally, we conclude the paper with remarks and references.

2. The Basic Method

In this section we concentrate on the case $v_1 = 0$. The general lacunary interpolation problem will be treated in Section 5. Following [3–7], we intend to use piecewise polynomials. In particular, let m be a positive integer with $m > v_p$. Let

$$s(x) = \begin{cases} s_1(x), & x \in I_1 \quad := [x_1, x_2) \\ \vdots & \\ s_{n-1}(x), & x \in I_{n-1} \quad := [x_{n-1}, x_n) \\ s_n(x), & x \in I_n \quad := [x_n, x_{n+1}], \end{cases} \tag{2.1}$$

where

$$s_i(x) = \sum_{k=0}^{m-1} \frac{s_i^k (x - x_i)^k}{k!}, \qquad i = 1, ..., n. \tag{2.2}$$

We now discuss how to select the coefficients $\{s_i^k\}$ so that s interpolates the data. Since s is a piecewise polynomial whose derivatives from the left and right at a knot x_i may not agree, we cannot enforce the interpolation conditions directly in the form (1.2). Instead, at each interior knot x_i, $2 \leqslant i \leqslant n$, we require that both polynomial pieces s_{i-1} and s_i interpolate the data at this knot. We are led to replace (1.2) by

$$D_{\mathrm{L}}^{v_j} s(x_i) = f_i^{v_j}, \qquad j = 1, ..., p \text{ and } i = 2, ..., n+1, \tag{2.3}$$

$$D_{\mathrm{R}}^{v_j} s(x_i) = f_i^{v_j}, \qquad j = 1, ..., p \text{ and } i = 1, ..., n, \tag{2.4}$$

where D_{L} and D_{R} denote the left and right derivatives, respectively.

Equations (2.3)–(2.4) are equivalent to

$$D_{\mathrm{L}}^{v_j} s_i(x_{i+1}) = f_{i+1}^{v_j}, \qquad j = 1, ..., p, \tag{2.5}$$

$$D_{\mathrm{R}}^{v_j} s_i(x_i) = f_i^{v_j}, \qquad j = 1, ..., p, \tag{2.6}$$

for $i=1,\ldots,n$. For each $1 \leqslant i \leqslant n$, this is a set of $2p$ equations to be satisfied by the mth order polynomial s_i. In order to solve these equations, we assume from now on that $m \geqslant 2p$.

We now discuss the problem of solving Eqs. (2.5)–(2.6). To this end it is convenient to introduce some additional notation. Let $I=\{v_1 < \cdots < v_p\}$, $J=\{v_{p+1} < \cdots < v_{2p}\}$ and $K=\{v_{2p+1} < \cdots < v_m\}$ be such that

$$M := \{0, 1,\ldots, m-1\} = I \cup J \cup K. \tag{2.7}$$

We do not assume that $I < J < K$ (i.e., v_m may be smaller than v_p, for example).

Fix $1 \leqslant i \leqslant n$. In view of (2.2), in order for s_i to satisfy (2.6) we must have

$$s_i^{v_j} = f_i^{v_j}, \qquad j=1,\ldots,p. \tag{2.8}$$

We have determined $\{s_i^v\}_{v \in I}$.

To satisfy (2.5), we must have

$$\sum_{k=v_j}^{m-1} \frac{s_i^k h_i^{k-v_j}}{(k-v_j)!} = f_{i+1}^{v_j}, \qquad j=1,\ldots,p, \tag{2.9}$$

where $h_i = x_{i+1} - x_i$. This is a system of p equations in the $m-p$ unknowns $\{s_i^v\}_{v \in J \cup K}$. If $m > 2p$ (i.e., $K \neq \varnothing$), we may choose $m-2p$ of these coefficients (say $\{s_i^v\}_{v \in K}$) arbitrarily. We elect to compute these coefficients by certain quadrature formulae. In particular, for each $v \in K$, we set

$$s_i^v = \sum_{\mu=1}^{n+1} \sum_{q=1}^{p} \alpha_{i,\mu}^{v,q} f_\mu^{v_q}, \tag{2.10}$$

where the α's are prescribed real numbers. We say more about the selection of these quadrature rules in Section 4.

Having chosen $\{s_i^v\}_{v \in I \cup K}$, it remains to find $\{s_i^v\}_{v \in J}$. To this end we go back to Eq. (2.9), which can now be regarded as a system of p equations in p unknowns. We may write this system in the form

$$A_i \begin{bmatrix} s_i^{v_{p+1}} \\ \vdots \\ s_i^{v_{2p}} \end{bmatrix} = \begin{bmatrix} f_{i+1}^{v_1} - \displaystyle\sum_{\substack{k=v_1 \\ k \notin J}}^{m-1} \frac{s_i^k h_i^{k-v_1}}{(k-v_1)!} \\ \vdots \\ f_{i+1}^{v_p} - \displaystyle\sum_{\substack{k=v_p \\ k \notin J}}^{m-1} \frac{s_i^k h_i^{k-v_p}}{(k-v_p)!} \end{bmatrix}, \tag{2.11}$$

where A_i is the $p \times p$ matrix with entries

$$(A_i)_{rc} = \frac{h_i^{\nu_{p+c} - \nu_r}}{(\nu_{p+c} - \nu_r)!}, \qquad \text{if } \nu_{p+c} \geqslant \nu_r$$
$$= 0, \qquad \text{otherwise.}$$

The following theorem shows that A_i is nonsingular if and only if

$$\nu_c \leqslant \nu_{p+c}, \qquad c = 1,..., p. \tag{2.12}$$

In addition, A_i is upper triangular provided

$$\nu_{p+c-1} < \nu_c, \qquad c = 2,..., p. \tag{2.13}$$

THEOREM 2.1. *The matrix A_i is nonsingular if and only if condition* (2.12) *holds.*

Proof. Clearly A_i is the minor

$$W = \begin{pmatrix} \nu_1 + 1,..., \nu_p + 1 \\ \nu_{p+1} + 1,..., \nu_{2p} + 1 \end{pmatrix}$$

of the matrix

$$W = (D^{i-1} x^{j-1}/(j-1)!)_{i,j=1}^m, \tag{2.14}$$

evaluated at $x = h_i$. We now show that (writing D for the determinant of W)

$$D\begin{pmatrix} i_1,..., i_p \\ j_1,..., j_p \end{pmatrix} \geqslant 0, \qquad \text{all} \quad \begin{matrix} 1 \leqslant i_1 < \cdots < i_p \leqslant m \\ 1 \leqslant j_1 < \cdots < j_p \leqslant m \end{matrix} \tag{2.15}$$

and all $p > 0$, and that moreover, strict positivity holds if and only if

$$i_\nu \leqslant j_\nu, \qquad \nu = 1,..., p. \tag{2.16}$$

This result is an extension of the fact that the powers $\{1, x,..., x^{m-1}\}$ form an order-complete extended Tchebycheff (OCET) system (cf. Theorem 3.7 of [12]).

First we show that if (2.16) fails, then the minor in (2.15) is zero. Suppose $i_\nu > j_\nu$ for some $1 \leqslant \nu \leqslant p$. Then this minor has zeros in the lower left rectangle in rows $\nu,..., p$ and columns $1,..., \nu$. This in turn implies that the first ν columns are linearly dependent, and hence the determinant is zero.

Now to establish that strict positivity holds in (2.15) under condition (2.16), we proceed in three steps. Since

$$D\begin{pmatrix} i,..., i+p-1 \\ j,..., j+p-1 \end{pmatrix} = D\begin{pmatrix} 1,..., p \\ j-i+1,..., j+p-i \end{pmatrix} > 0$$

for all p and all $1 \leqslant i \leqslant j \leqslant m-p+1$ by the OCET property, we have established the result when there are no gaps in the sequences $i_1,..., i_p$ and $j_1,..., j_p$.

We now proceed by induction on the number of gaps in the sequence of i's. Suppose $1 \leqslant i_1 < \cdots < i_p \leqslant m$ has g gaps and that (2.16) holds. Let μ be the first missing index, and let q be such that $i_{q-1} \leqslant \mu < i_q$. Then (2.16) implies

$$\begin{aligned} & i_1 < j \\ & \vdots \\ & i_{q-1} < j+q-2 \\ & i_q \leqslant j+q-1 \\ & \vdots \\ & i_p \leqslant j+p-1 \end{aligned} \qquad \text{and so} \qquad \begin{aligned} & i_2 \leqslant j \\ & \vdots \\ & i_{q-1} \leqslant j+q-3 \\ & \mu \leqslant j+q-2 \\ & i_q \leqslant j+q-1 \\ & \vdots \\ & i_p \leqslant j+p-1. \end{aligned} \tag{2.17}$$

Now by a well-known matrix identity (cf. Remark 3.2 of [12]),

$$\begin{aligned} & D\begin{pmatrix} i_1,..., i_p \\ j,..., j+p-1 \end{pmatrix} D\begin{pmatrix} i_2,..., i_{p-1}, \mu \\ j,..., j+p-2 \end{pmatrix} \\ & \quad = D\begin{pmatrix} i_1,..., i_{p-1}, \mu \\ j,..., j+p-1 \end{pmatrix} D\begin{pmatrix} i_2,..., i_p \\ j,..., j+p-2 \end{pmatrix} \\ & \qquad - D\begin{pmatrix} i_2,..., i_p, \mu \\ j,..., j+p-1 \end{pmatrix} D\begin{pmatrix} i_1,..., i_{p-1} \\ j,..., j+p-2 \end{pmatrix}. \end{aligned}$$

The determinant of interest is the first on the left. All other determinants correspond to minors of size $p-1$ or to i sequences with one less gap $g-1$. By (2.17) and the inductive hypothesis that the result holds for $p-1$ and for p with $g-1$ gaps, we get positivity for p and g gaps.

To complete the proof, we have to allow gaps into the j sequence. This we do by induction on the number of gaps g in the sequence $1 \leqslant j_1 < \cdots < j_p \leqslant m$, using the same kind of argument as above. ∎

We may summarize the discussion in this section in the following algorithm to compute a spline (2.1) solving the interpolation conditions (2.3)–(2.4).

ALGORITHM 2.2. Suppose I, J, and K are such that (2.12) holds.

For $i=1, n$

(1) For $j=1$ step 1 until p

$$s_i^{\nu_j} = f_i^{\nu_j}.$$

(2) For $j=2p+1$ step 1 until m

$$s_i^{\nu_j} = \sum_{\mu=1}^{n+1} \sum_{q=1}^{p} \alpha_{i,\mu}^{\nu_j,q} f_\mu^{\nu_q}.$$

(3) Solve (2.11) for $s_i^{\nu_{p+1}},..., s_i^{\nu_{2p}}$.

Discussion. The exact nature of this algorithm depends on the choice of I, J, K, and the quadrature coefficients $\{\alpha_{i,\mu}^{\nu_j,q}\}$. We say more about the choice of α's in Section 4. If $m=2p$, then step (2) can be eliminated altogether. Moreover, if J and K can be chosen so that both (2.12) and (2.13) hold, then step (3) can be accomplished by back substitution, and no linear system solver is needed. In this case step (3) can be written as

(3)′ For $j=p$ step -1 until 1

$$s_i^{\nu_{j+p}} = (\nu_{j+p}-\nu_j)!\, h_i^{\nu_j-\nu_{j+p}} \left(f_{i+1}^{\nu_j} - \sum_{\substack{k=\nu_j \\ k \notin J}}^{m-1} \frac{s_i^k h_i^{k-\nu_j}}{(k-\nu_j)!} - \sum_{k=j+p+1}^{2p} \frac{s_i^{\nu_k} h_i^{\nu_k-\nu_j}}{(\nu_k-\nu_j)!} \right). \quad \blacksquare$$

3. ERROR BOUNDS

In this section we investigate how well the lacunary interpolation method of Section 2 performs in approximating smooth functions. In particular, we suppose that the data are given by

$$f_i^{\nu_j} = D^{\nu_j} f(x_i), \qquad j=1,..., p \text{ and } i=1,..., n+1, \tag{3.1}$$

where f is a smooth function. Then if s is our interpolating spline, we are interested in bounds on $f-s$ and its derivatives.

Our results here will deal with functions f taken from $C^{\sigma-1}[x_1, x_{n+1}]$ or

$$L_p^\sigma[x_1, x_{n+1}] = \{f \in C^{\sigma-1}[x_1, x_{n+1}]: f^{(\sigma)} \in L_p[x_1, x_{n+1}]\} \tag{3.2}$$

We shall measure $f-s$ using the usual q-norms. When it comes to measuring the derivatives we have to adopt the usual convention for dealing with functions (such as s) which are only piecewise smooth: we define

$$\|D^j(f-s)\|_q = \left(\sum_{i=1}^{n} \|D^j(f-s)\|^q_{L_q[x_i, x_{i+1}]} \right)^{1/q}. \tag{3.3}$$

The basis for our error analysis will be the usual Taylor expansion. Fix $v_p < \sigma \leqslant m$. If $f \in C^{\sigma-1}[x_i, x_{i+1}]$, then for $x_i \leqslant x \leqslant x_{i+1}$.

$$f^{(d)}(x) = \sum_{k=d}^{\sigma-2} \frac{f_i^k (x-x_i)^{k-d}}{(k-d)!} + \frac{f^{(\sigma-1)}(\xi_{id})(x-x_i)^{\sigma-d-1}}{(\sigma-d-1)!}, \tag{3.4}$$

where $f_i^k = f^{(k)}(x_i)$, $k=0,..., \sigma-2$ and $x_i \leqslant \xi_{id} \leqslant x$. By (2.2), for $x_i \leqslant x < x_{i+1}$,

$$s^{(d)}(x) = \sum_{k=d}^{m-1} \frac{s_i^k (x-x_i)^{k-d}}{(k-d)!}. \tag{3.5}$$

Thus

$$\begin{aligned} |D^d(f-s)(x)| \leqslant & \sum_{k=d}^{\sigma-1} \frac{|f_i^k - s_i^k|\, h_i^{k-d}}{(k-d)!} \\ & + \frac{|f^{(\sigma-1)}(\xi_{i,d}) - f_i^{\sigma-1}|\, h_i^{\sigma-d-1}}{(\sigma-d-1)!} + \sum_{k=\sigma}^{m-1} \frac{|s_i^k|\, h_i^{k-d}}{(k-d)!}, \end{aligned} \tag{3.6}$$

where $h_i = x_{i+1} - x_i$.

We intend to estimate (3.6) in terms of the modulus of continuity of $f^{(\sigma-1)}$. The second term in (3.6) can be bounded directly in terms of $\omega(f^{(\sigma-1)}; h)$. Clearly, the size of the first sum will depend on how accurate our estimates s_i^k for f_i^k are. These will in turn depend on the accuracy of the quadrature formulae used in step (2) of Algorithm 2.2.

Suppose now that the α's in the quadrature formula are chosen so that for each $1 \leqslant i \leqslant n$, and any $f \in C^{\sigma-1}[x_1, x_{n+1}]$,

$$\begin{aligned} Ch^{\sigma-v_j-1}\omega(f^{(\sigma-1)}; h) &\geqslant |f_i^{v_j} - s_i^{v_j}|, && \text{if } 0 \leqslant v_i \leqslant \sigma-1 \\ &\geqslant |s_i^{v_j}|, && \text{if } \sigma \leqslant v_j \leqslant m-1 \end{aligned} \tag{3.7}$$

for $j = 2p+1,..., m$, where C is a constant independent of f and $h = \max_{1 \leqslant i \leqslant n} h_i$. In Section 4 below we show how to construct quadrature formulae yielding this order of approximation. The hypothesis (3.7) takes care of the νth derivatives for $\nu \in K$. The following lemma deals with the remaining derivatives ($\nu \in I \cup J$).

LEMMA 3.1. *Suppose Algorithm* 2.2 *is designed so that* (3.7) *holds. Then for any* $f \in C^{\sigma-1}[x_1, x_{n+1}]$ *with* $\nu_p < \sigma \leqslant m$,

$$\begin{aligned} Ch^{\sigma-\nu-1}\omega(f^{(\sigma-1)}; h) &\geqslant |f_i^\nu - s_i^\nu|, && \text{if } 0 \leqslant \nu \leqslant \sigma - 1 \\ &\geqslant |s_i^\nu|, && \text{if } \sigma \leqslant \nu \leqslant m-1, \end{aligned} \tag{3.8}$$

$i = 1,..., n$, *where C is a constant.*

Proof. The estimate (3.8) is trivial for $\nu \in I$ as $f_i^\nu = s_i^\nu$ and $\nu \leqslant \sigma - 1$ for these ν. The assumption (3.7) takes care of $\nu \in K$. It remains to deal with $\nu \in J$. We make use of Eq. (2.11). First we note that using (3.4) with $x = x_i + h_i$, the jth component of the right-hand side of (2.11) becomes

$$\begin{aligned} r_i^j := \sum_{\substack{k=\nu_j \\ k \notin J}}^{\sigma-2} \frac{(f_i^k - s_i^k)\, h_i^{k-\nu_j}}{(k-\nu_j)!} + \frac{f^{\sigma-1}(\xi_i)\, h_i^{\sigma-1-\nu_j}}{(\sigma-1-\nu_j)!} \\ + \sum_{\substack{k=\nu_j \\ k \in J}}^{\sigma-2} \frac{f_i^k h_i^{k-\nu_j}}{(k-\nu_j)!} - \sum_{\substack{k=\sigma-1 \\ k \notin J}}^{m-1} \frac{s_i^k\, h_i^{k-\nu_j}}{(k-\nu_j)!}. \end{aligned} \tag{3.9}$$

Now fix $1 \leqslant l \leqslant p$. We establish (3.8) for $\nu = \nu_{l+p}$. By Cramer's rule,

$$s_i^{\nu_{l+p}} = \frac{D_l(\bar{r})}{D}, \tag{3.10}$$

where $D = \det A_i$, $\bar{r} = (r_i^1,..., r_i^p)^T$, and $D_l(\bar{r})$ denotes the determinant of the matrix A_i with its lth column replaced by $\bar{r}$. By the multi-linear nature of determinants, for general $\bar{g} = (g_1,..., g_p)$

$$\frac{D_l(\bar{g})}{D} = \sum_{j=1}^{p} g_j C_{lj}, \tag{3.11}$$

where the "cofactors" $C_{lj} = O(h_i^{\nu_j - \nu_{p+l}})$. Expanding (3.10) out using (3.11), we obtain

$$\begin{aligned} s_i^{\nu_{l+p}} = \sum_{j=1}^{p} C_{lj} \Bigg(\sum_{\substack{k=\nu_j \\ k \notin J}}^{\sigma-2} \frac{(f_i^k - s_i^k)\, h_i^{k-\nu_j}}{(k-\nu_j)!} + \frac{f^{(\sigma-1)}(\xi_i)\, h_i^{\sigma-\nu_j-1}}{(\sigma-1-\nu_j)!} \\ + \sum_{\substack{k=\nu_j \\ k \in J}}^{\sigma-2} \frac{f_i^k h_i^{k-\nu_j}}{(k-\nu_j)!} - \sum_{\substack{k=\sigma-1 \\ k \notin J}}^{m-1} \frac{s_i^k\, h_i^{k-\nu_j}}{(k-\nu_j)!} \Bigg). \end{aligned} \tag{3.12}$$

We investigate the size of each of these four major terms. Using (3.7), we see that

$$|f_i^k - s_i^k|\, h_i^{k-\nu_j} C_{lj} = O(h_i^{\sigma-k-1} \cdot h_i^{k-\nu_j} \cdot h_i^{\nu_j-\nu_{p+1}}\omega)$$
$$= O(h_i^{\sigma-\nu_{p+1}}\,\omega),$$

where $\omega = \omega(f^{(\sigma-1)}; h)$. Similarly, the terms in the fourth sum (except for $k=\sigma-1$) can be estimated in the same way. If $\sigma - 1 \notin J$, we may treat the term $k=\sigma-1$ by adding and subtracting $\sum_{j=1}^{p} C_{lj} f_i^{\sigma-1} h_i^{\sigma-\nu_j-1}/(\sigma-\nu_j-1)!$. In this case we can now combine this expression with the second term to get $O(h^{\sigma-1-\nu_{p+l}}\omega)$. Suppose now $\sigma-1 \in J$. We now consider three cases.

Case 1. $0 \leqslant \nu_{l+p} < \sigma - 1$. In this case, it is easy to see that all terms in the third sum are zero except for $k = \nu_{l+p}$ (indeed D_l will contain two identical columns). The remaining term becomes

$$\sum_{j=1}^{p} C_{lj} \frac{f_i^k h_i^{k-\nu_j}}{(k-\nu_j)!} = f_i^{\nu_{l+p}}.$$

Since the second term can be estimated by adding $-\sum_{j=1}^{p} C_{lj} f_i^{\sigma-1} h_i^{\sigma-1-\nu_j}/(\sigma-1-\nu_j)! = 0$ and using

$$|f^{(\sigma-1)}(\xi_i) - f_i^{\sigma-1}| \leqslant c\omega(f^{(\sigma-1)}; h),$$

we end up with

$$|f_i^{\nu_{l+p}} - s_i^{\nu_{l+p}}| = O(h^{\sigma-1-\nu_{l+p}}\omega).$$

Case 2. $\nu_{l+p} = \sigma - 1$. In this case we add and subtract $f_i^{\sigma-1}$ and get the same estimate.

Case 3. $\sigma - 1 < \nu_{l+p}$. Here we need only estimate the second term as in Case 1 to obtain

$$|s_i^{\nu_{l+p}}| = O(h_i^{\sigma-\nu_{l+p}-1}\omega).$$

This completes the proof. ∎

We are ready for the main theorem of this section.

THEOREM 3.2. *Suppose Algorithm* 2.2 *is designed so that* (3.7) *holds. Let* $\nu_p < \sigma \leqslant m$, *and let* $h = \max_{1 \leqslant i \leqslant n}(x_{i+1} - x_i)$. *Then for every* $f \in C^{\sigma-1}[x_1, x_{n+1}]$, *and every* $0 \leqslant d \leqslant \sigma - 1$,

$$\begin{aligned} C^* h^{\sigma-d-1}\omega(f^{(\sigma-1)}; h) &\geqslant \|D^d(f-s)\|_\infty, \qquad \text{if } 0 \leqslant d \leqslant \sigma-1 \\ &\geqslant \|D^d s\|_\infty, \qquad \text{if } \sigma \leqslant d \leqslant m-1, \end{aligned} \tag{3.13}$$

where C^* *is a constant.*

Proof. Substituting (3.8) in (3.6), we obtain

$$|D^d(f-s)(x)| \leqslant \left(C \sum_{k=d}^{m-1} \frac{h^{\sigma-k-1}h^{k-d}}{(k-d)!} + \frac{h^{\sigma-d-1}}{(\sigma-d-1)!} \right) \omega(f^{(\sigma-1)}; h)$$
$$\leqslant C^* h^{\sigma-d-1} \omega(f^{(\sigma-1)}; h) \tag{3.14}$$

for $x_i \leqslant x < x_{i+1}$. The result follows. ∎

To establish a similar result for functions in the class $L_p^\sigma[x_1, x_{n+1}]$, suppose now that the α's in the quadrature formulae defining Algorithm 2.2 are chosen so that

$$Ch^{\sigma-v_j-1/p} \|f^{(\sigma)}\|_{L_p[J_i]} \geqslant |f_i^{v_j} - s_i^{v_j}|, \qquad \text{if } 0 \leqslant v_j \leqslant \sigma - 1$$
$$\geqslant |s_i^{v_j}|, \qquad \text{if } \sigma \leqslant v_j \leqslant m-1, \tag{3.15}$$

for $j = 2p+1, ..., m$, where C is a constant independent of f, and J_i is an interval containing x_i.

We have the following analog of Lemma 3.1.

LEMMA 3.3. *Suppose Algorithm* 2.2 *is designed so that* (3.15) *holds. Then for any* $f \in L_p^\sigma[x_1, x_{n+1}]$ *with* $v_p \leqslant \sigma \leqslant m$,

$$C^* h^{\sigma-v-1/p} \|f^{(\sigma)}\|_{L_p[J_i]} \geqslant |f_i^v - s_i^v|, \qquad \text{if } 0 \leqslant v \leqslant \sigma - 1$$
$$\geqslant |s_i^v|, \qquad \text{if } \sigma \leqslant v \leqslant m-1, \tag{3.16}$$

$i = 1, ..., n$, *where* C^* *is a constant and* J_i *is the interval in* (3.15).

Proof. Since $f \in L_p^\sigma[x_1, x_{n+1}]$ implies $f \in C^{\sigma-1}[x_1, x_{n+1}]$ we may use the proof of Lemma 3.1 along with

$$|f^{(\sigma-1)}(x_i) - f^{(\sigma-1)}(\eta_{i,2p})| \leqslant \int_{\eta_{i,2p}}^{x_i} |f^{(\sigma)}(t)|\, dt \leqslant \|f^{(\sigma)}\|_{L_p[x_i, x_{i+1}]} h_i^{1-1/p}. \quad ∎$$

We now give the analog of Theorem 3.2 for L_p^σ functions.

THEOREM 3.4. *Suppose Algorithm* 2.2 *is designed so that* (3.15) *holds. Let* $v_p \leqslant \sigma \leqslant m$ *and let* $h = \max_{1 \leqslant i \leqslant n}(x_{i+1} - x_i)$. *Let* $1 \leqslant p \leqslant q \leqslant \infty$. *Then for every* $f \in L_p^\sigma[x_1, x_{n+1}]$ *and every* $0 \leqslant d \leqslant \sigma - 1$,

$$C^* h^{\sigma-d-1/p+1/q} \omega(f^{(\sigma)}; h)_p \geqslant \|D^d(f-s)\|_q, \qquad \text{if } 0 \leqslant d \leqslant \sigma - 1$$
$$\geqslant \|D^d s\|_q, \qquad \text{if } \sigma \leqslant d \leqslant m-1. \tag{3.17}$$

Proof. The second term in (3.6) can be treated in the same way as the similar expression was in the proof of Lemma 3.3. Then using (3.16) in (3.6) just as in the proof of Theorem 3.2, we get

$$|D^d(f-s)(x)| \leqslant Ch^{\sigma-d-1/p} \|f^{(\sigma)}\|_{L_p[J_i]} \tag{3.18}$$

for $x_i \leqslant x \leqslant x_{i+1}$, where J_i is as in (3.15). Now taking the qth power and integrating over $[x_i, x_{i+1}]$, we get

$$\int_{x_i}^{x_{i+1}} |D^d(f-s)(x)|^q \, dx \leqslant C^q h^{q(\sigma-d-1/p)+1} \|f^{(\sigma)}\|^q_{L_p[J_i]}. \tag{3.19}$$

Summing over $i=1,..., n$ and taking the qth root, we obtain

$$\|D^d(f-s)\|_{L_q[x_1,x_{n+1}]} \leqslant Ch^{\sigma-d-1/p+1/q} \left(\sum_{i=1}^{n} \|f^{(\sigma)}\|^q_{L_p[J_i]} \right)^{1/q}.$$

Using Jensen's inequality (see [12, p. 205]),

$$\left(\sum_{i=1}^{n} \|f^{(\sigma)}\|^q_{L_p[J_i]} \right)^{1/q} \leqslant \left(\sum_{i=1}^{n} \|f^{(\sigma)}\|^p_{L_p[J_i]} \right)^{1/p} \leqslant L \|f^{(\sigma)}\|_{L_p[x_1,x_{n+1}]}.$$

We conclude that

$$\|D^d(f-s)\|_{L_q[x_1,x_{n+1}]} \leqslant C^* h^{\sigma-d-1/p+1/q} \|f^{(\sigma)}\|_{L_p[x_1,x_{n+1}]}. \tag{3.20}$$

The quantity $\|f^{(\sigma)}\|$ on the right-hand side of (3.20) can be replaced by $\omega(f^{(\sigma)}; h)$ by using the K-functional in exactly the same way as in the proof of Theorem 6.25 in [12]. ∎

4. Choice of Quadrature Formulae

Our aim in this section is to show how to construct quadrature formulae for estimating derivatives of smooth functions which yield error bounds of the type (3.7). We begin by considering the following problem.

Problem 4.1. Let $x_1 < \cdots < x_{n+1}$. Let $1 \leqslant i \leqslant n+1$, and suppose m and $v < m$ are positive integers. Find $\{\alpha_j\}_1^{n+1}$ such that the quadrature formula

$$Qf = \sum_{j=1}^{n+1} \alpha_j f(x_j) \tag{4.1}$$

satisfies

$$|f^{(v)}(x_i) - Qf| \leqslant Ch^{\sigma-v-1} \omega(f^{(\sigma-1)}; h) \tag{4.2}$$

if $f \in C^{\sigma-1}[x_1, x_{n+1}]$ and $\nu \leqslant \sigma - 1 \leqslant m-1$ and

$$|Qf| \leqslant Ch^{\sigma-\nu-1}\omega(f^{(\sigma-1)}; h) \tag{4.3}$$

if $f \in C^{\sigma-1}[x_1, x_{n+1}]$ and $1 \leqslant \sigma \leqslant \nu$. Here $h = \max_{1 \leqslant i \leqslant n}(x_{i+1} - x_i)$, and C should be a constant, independent of f, σ, and h.

Since $\omega(f^{(\sigma-1)}; h) = 0$ if f is a polynomial of degree $\sigma - 1$, in order for (4.2) to hold in the extreme case $\sigma = m$, we must have

$$f^{(\nu)}(x_i) = Qf \qquad \text{whenever } f \text{ is a polynomial of degree } m-1. \tag{4.4}$$

This is a set of m conditions. Thus it is natural to try to find Qf in the form (4.1), but with only m non-zero coefficients. Since Qf is to approximate $f^{(\nu)}$ at x_i, it is also reasonable to make Qf involve values of f at points as close to x_i as possible.

The above considerations suggest that we try to solve Problem 4.1 by taking

$$Qf = \sum_{j=l_i}^{l_i+m-1} \alpha_j f(x_j), \tag{4.5}$$

where $\{\alpha_j\}$ satisfy the linear system

$$\sum_{j=l_i}^{l_i+m-1} \alpha_j \phi_k(x_j) = \phi_k^{(\nu)}(x_i), \qquad k = 1,..., m, \tag{4.6}$$

with $\{\phi_1,..., \phi_m\} = \{1, (x - x_i),..., (x - x_i)^{m-1}\}$, and where

$$l_i = \min\{j\colon 1 \leqslant j \leqslant n - m + 2 \text{ and } j \geqslant i - m/2\}. \tag{4.7}$$

The choice of l_i as in (4.7) assures that the sample points in (4.5) are as close to centered about x_i as possible.

The system (4.6) can be written in the form

$$B\alpha = r, \tag{4.8}$$

where $\alpha = (\alpha_{l_i},..., \alpha_{l_i+m-1})^{\mathrm{T}}$, $r = \nu!\, e_\nu$, e_ν, is an m-vector with all zero entries except for a 1 in the νth position, and where B is the $m \times m$ matrix with $B_{rc} = (x_{l_i+c-1} - x_i)^{r-1}$.

Clearly B is a van der Monde matrix, and hence is non-singular. Thus, the coefficients α are uniquely determined. We can now give an error bound for the resulting quadrature formula.

THEOREM 4.2. *Given $x_1 < \cdots < x_{n+1}$ and positive integers i, ν, m with*

$1 \leqslant i \leqslant n+1$, $0 < \nu < m$, *let* l_i *be as in* (4.7) *and let* $\alpha = (\alpha_{l_i}, ..., \alpha_{l_i+m-1})^T$ *be the solution of* (4.8). *Then the quadrature formula*

$$Qf = \sum_{j=l_i}^{l_i+m-1} \alpha_j f(x_j) \tag{4.9}$$

satisfies (4.2) *and* (4.3).

Proof. By Cramer's rule, for each $1 \leqslant c \leqslant m$,

$$\alpha_{l_i+c} = (-1)^{c+\nu} \nu! \frac{\det B_c}{\det B}, \tag{4.10}$$

where B_c is the $m-1 \times m-1$ matrix obtained from B by deleting the νth row and cth column. Now by the multilinear nature of the determinants

$$\det B_c = C_1 h^{m(m-1)/2-\nu}$$
$$\det B = C_2 h^{m(m-1)/2}.$$

We conclude that $|\alpha_{l_i+c-1}| \leqslant C_3 h^{-\nu}$, $c = 1, ..., m$, and thus

$$\|Q\| = \sup_{\substack{f \neq 0 \\ f \in C[x_1, x_{n+1}]}} \frac{|Qf|}{\|f\|_\infty} \leqslant \sum_{c=1}^{m} |\alpha_{l_i+c-1}| \leqslant C_4 h^{-\nu}. \tag{4.11}$$

By an extension of a result of Whitney (cf. Lemma 4.4 below), there exists a constant C_5 depending only on m such that if $f \in C^{\sigma-1}[x_1, x_{n+1}]$ with $1 \leqslant \sigma \leqslant m$, then there exists a polynomial p_f of degree $m-1$ with

$$\begin{aligned} C_5 h^{\sigma-j-1} \omega(f^{(\sigma-1)}; h) &\geqslant \|D^j(f-p_f)\|, \qquad j = 0, 1, ..., \sigma-1 \\ &\geqslant \|D^j p_f\|, \qquad j = \sigma, ..., m-1. \end{aligned} \tag{4.12}$$

Now let $\nu \leqslant \sigma - 1 \leqslant m-1$ and suppose $f \in C^{\sigma-1}[x_1, x_{n+1}]$. Then

$$|f^{(\nu)}(x_i) - Qf| \leqslant |f^{(\nu)}(x_i) - p_f^{(\nu)}(x_i)| + |p_f^{(\nu)}(x_i) - Qp_f| + |Q(p_f - f)|.$$

The second term is zero since Q is exact for polynomials of degree $m-1$. Using (4.11) and (4.12), we get

$$\begin{aligned} |f^{(\nu)}(x_i) - Qf| &\leqslant \|f^{(\nu)} - p_f^{(\nu)}\| + \|Q\| \, \|f - p_f\| \\ &\leqslant C_6 h^{\sigma-\nu-1} \omega(f^{(\sigma-1)}; h). \end{aligned}$$

We have established (4.2).

Finally suppose $1 \leqslant \sigma \leqslant \nu$ and $f \in C^{\sigma-1}[x_1, x_{n+1}]$. Then

$$|Qf| \leqslant |Q(f-p_f)| + |Qp_f - p_f^{(\nu)}(x_i)| + |p_f^{(\nu)}(x_i)|.$$

As before the second term is zero, and combining (4.11) and (4.12) yields (4.3). ∎

Our next theorem gives error bounds for the quadrature formula Q in (4.9) for functions in the Sobolev space $L_p^\sigma[x_1, x_{n+1}]$.

THEOREM 4.3. *Let $1 \leqslant p \leqslant \infty$, and let Q_f be the quadrature formula of Theorem 4.2 for approximating $f^{(v)}(x_i)$. Then for all $f \in L_p^\sigma[x_1, x_{n+1}]$, Q satisfies*

$$|f^{(v)}(x_i) - Qf| \leqslant Ch^{\sigma - v - 1/p}\omega(f^{(\sigma)}; h)_p \qquad \text{if } v < \sigma \leqslant m \tag{4.13}$$

and

$$|Qf| \leqslant Ch^{\sigma - v - 1/p}\omega(f^{(\sigma)}; h)_p \qquad \text{if } 1 \leqslant \sigma \leqslant v. \tag{4.14}$$

Here C is a constant independent of f, σ, and h, and $\omega(\cdot)_p$ is the L_p-modulus of continuity (see [12]).

Proof. The proof proceeds exactly as in Theorem 4.2 but using the fact (cf. Lemma 4.4 below) that there exists a constant C_1 depending only on m and p such that if $f \in L_p^\sigma[x_1, x_{n+1}]$ with $1 \leqslant \sigma \leqslant m$, then there exists a polynomial p_f of degree m with

$$\begin{aligned} C_1 h^{\sigma - 1/p - j}\omega(f^{(\sigma)}; h)_p &\geqslant \|D^j(f - p_f)\|_\infty, & j &= 0, \ldots, \sigma - 1 \\ &\geqslant \|D^j p_f\|_\infty, & j &= \sigma, \ldots, m - 1. \end{aligned} \tag{4.15}$$ ∎

LEMMA 4.4. *Let $1 \leqslant \sigma \leqslant m$ and $a < b$. Then there exists a constant C_1 (depending only on m) such that for all $f \in C^{\sigma-1}[a, b]$, there exists a polynomial p_f of degree $m - 1$ with*

$$\begin{aligned} C_1 h^{\sigma - j - 1}\omega(f^{(\sigma-1)}; h) &\geqslant \|D^j(f - p_f)\|, & 0 &\leqslant j \leqslant \sigma - 1 \\ &\geqslant \|D^j p_f\|, & \sigma &\leqslant j \leqslant m - 1, \end{aligned} \tag{4.16}$$

where $h = |b - a|$. Moreover, given $1 \leqslant p \leqslant \infty$, there exists a constant C_2 (depending only on m and p) such that for all $f \in L_p^\sigma[a, b]$, there exists a polynomial $\tilde{p}_f$ of degree $m - 1$ with

$$\begin{aligned} C_2 h^{\sigma - j + 1/q - 1/p}\omega(f^{(\sigma)}; h)_{L_p[a,b]} &\geqslant \|D^j(f - \tilde{p}_f)\|_q, & 0 &\leqslant j \leqslant \sigma - 1 \\ &\geqslant \|D^j \tilde{p}_f\|_q, & \sigma &\leqslant j \leqslant m - 1, \end{aligned} \tag{4.17}$$

for all $1 \leqslant q \leqslant \infty$.

Proof. This result is an extension of Theorems 3.19 and 3.20 in [12]. It will suffice to sketch the changes needed in the proofs presented there. We consider only $p = \infty$. The case $1 \leqslant p < \infty$ is similar.

Following the proofs of Theorems 2.66, 3.18, and 3.19, let $g=f^{(\sigma-1)}\in C[a,b]$. Applying the Whitney extension theorem let $\tilde{g}$ be the extension to $C[a, 2b-a]$ with $\omega(\tilde{g};t)\leqslant C\omega(g;t)$. Now associated with $\tilde{g}$, let $\bar{g}$ be the Steklov average of $\tilde{g}$ of order $m-\sigma+1$ with $\|g-\bar{g}\|\leqslant C\omega_{m-\sigma+1}(g;h)$. Then it is easy to see that $\|D^j\bar{g}\|\leqslant Ch^{-(m-\sigma+1)}\omega_{m-\sigma+1}(g^{(j-m+\sigma-1)};h)$ for $j=0,..., m-\sigma+1$. Now the polynomial

$$q(x)=\sum_{j=0}^{m-\sigma}\frac{\bar{g}^{(j)}(a)(x-a)^j}{j!} \tag{4.18}$$

satisfies $\|g-q\|\leqslant C\omega_{m-\sigma+1}(g;h)$ and $\|D^jq\|\leqslant C\omega_{m-\sigma+1}(g^{(j-m+\sigma-1)};h)$, $j=0,..., m-\sigma+1$. Applying an inequality on moduli of smoothness (cf. (2.119) of [12]), we get $\|D^jq\|\leqslant h^jC\omega(g;h), j=0,..., m-\sigma$. Now let

$$p_f(x)=\sum_{j=0}^{\sigma-2}\frac{f^{(j)}(a)(x-a)^j}{j!}+\int_a^b\frac{(x-a)_+^{\sigma-2}\,q(y)\,dy}{(\sigma-2)!}. \tag{4.19}$$

Clearly p_f is a polynomial of degree $m-1$ with $D^{\sigma-1}p_f=q$ and $p_f^{(j)}(a)=f_a^{(j)}(a)$, $j=0,..., \sigma-2$. Then it follows that $\|D^j(f-p_f)\|\leqslant h^{\sigma-j-1}\|D^{\sigma-1}(f-p_f)\| = h^{\sigma-j-1}\|g-q\| \leqslant Ch^{\sigma-j-1}\,\omega(g;h) = Ch^{\sigma-j-1}\,\omega(f^{(\sigma-1)};h)$, for $j=0, 1,..., \sigma-1$. This is the first part of (4.16).

For the second part of (4.16), let $\sigma\leqslant j\leqslant m-1$. Then

$$\|D^jp_f\|=\|D^{j-\sigma+1}D^{\sigma-1}p_f\|=\|D^{j-\sigma+1}q\|.$$

Now by the Markov inequality (cf. Theorem 3.3 of [12]),

$$\|D^{j-\sigma+1}q\|\leqslant h^{\sigma-j-1}\|q\|\leqslant Ch^{\sigma-j-1}\omega(g;h)=Ch^{\sigma-j-1}\omega(f^{(\sigma-1)};h).$$

This completes the proof. ∎

We conclude this section with a simple but useful observation. Suppose Q is a quadrature formula for estimating $g^{(\nu-\mu)}(x_i)$ of the form

$$Qg=\sum_{j=1}^{n+1}\alpha_j(x_j) \tag{4.20}$$

such that if $g\in C^{\sigma-\mu-1}[x_1, x_{n+1}]$,

$$\begin{aligned} Ch^{\sigma-\nu-1}\omega(g^{(\sigma-\mu-1)};h) &\geqslant |g^{(\nu-\mu)}(x_i)-Qg|, && \text{if } \nu-\mu\leqslant\sigma-1\\ &\geqslant |Qg|, && \text{if } \sigma\leqslant\nu-\mu. \end{aligned} \tag{4.21}$$

Then the quadrature formula

$$Qf=\sum_{j=1}^{n+1}\alpha_jf^{(\mu)}(x_j) \tag{4.22}$$

can be used to estimate $f^{(\nu)}(x_i)$. Moreover, taking $g = f^{(\mu)}$ in (4.20)–(4.21), we see that if $f \in C^{\sigma-1}[x_1, x_{n+1}]$, then Qf satisfies properties (4.2)–(4.3).

5. The Case $\nu_1 > 0$

So far in this paper we have concentrated our attention on the lacunary interpolation Problem 1.1 defined by $I = (\nu_1, ..., \nu_p)$ with $\nu_1 = 0$. In this section we consider the case where $\nu_1 > 0$.

We begin by observing that when $\nu_1 > 0$, if Problem 1.1 has a solution, then it will have a whole ν_1-parameter family of solutions. Indeed, if s satisfies the interpolation conditions (1.2), then so does $s + p$, where p is any polynomial of degree $\nu_1 - 1$. To fix p, it is natural to add ν_1 conditions to the original interpolation conditions.

THEOREM 5.1. *Let Δ, I, and $\{f_i^{\nu_j}\}_{j=1,i=1}^{p,n+1}$ be as in Problem 1.1. In addition, let $\{f_1^{\nu}\}_{\nu=0}^{\nu_1-1}$ be given real numbers. Let $m > \nu_p$, and let*

$$s(x) = \begin{cases} \displaystyle\sum_{k=0}^{m-\nu_1-1} \frac{s_1^k (x - x_1)^k}{k!}, & x \in [x_1, x_2) \\ \vdots & \\ \displaystyle\sum_{k=0}^{m-\nu_1-1} \frac{s_n^k (x - x_n)^k}{k!}, & x \in [x_n, x_{n+1}] \end{cases} \tag{5.1}$$

be the $(m - \nu_1)$th order piecewise polynomial produced by Algorithm 2.2 and solving the interpolation problem

$$\begin{aligned} D_{\mathrm{R}}^{(\nu_j - \nu_1)} s(x_i) &= f_i^{\nu_j}, & j &= 1, ..., p \text{ and } i = 2, ..., n+1 \\ D_{\mathrm{L}}^{(\nu_j - \nu_1)} s(x_i) &= f_i^{\nu_j}, & j &= 1, ..., p \text{ and } i = 1, ..., n. \end{aligned} \tag{5.2}$$

Let

$$S(x) = \begin{cases} S_1(x) := \displaystyle\sum_{k=0}^{m-1} \frac{S_1^k (x - x_1)^k}{k!}, & x \in [x_1, x_2) \\ \vdots & \\ S_n(x) := \displaystyle\sum_{k=0}^{m-1} \frac{S_n^k (x - x_n)^k}{k!}, & x \in [x_n, x_{n+1}], \end{cases} \tag{5.3}$$

where

$$S_i^k = \begin{cases} S_{i-1}^{(k)}(x_i), & k = 0, ..., \nu_1 - 1 \\ S_i^{k-\nu_1}, & k = \nu_1, ..., m-1 \end{cases} \tag{5.4}$$

and

$$S_0^k(x_1) = f_1^k, \qquad k = 0,..., \nu_1 - 1.$$

Then S *is a piecewise polynomial of order* m *which solves Problem* 1.1, *and which satisfies the initial conditions*

$$S^{(\nu)}(x_1) = f_1^\nu, \qquad \nu = 0,..., \nu_1 - 1. \tag{5.5}$$

Proof. Clearly S satisfies (5.5), and $S^{(\nu_1)} = s$. It follows immediately that for all i,

$$D_{\mathrm{L}}^{\nu_j} S(x_i) = D_{\mathrm{L}}^{\nu_j - \nu_i} s(x_i) \qquad \text{and} \qquad D_{\mathrm{R}}^{\nu_j} S(x_i) = D_{\mathrm{R}}^{\nu_j - \nu_1} s(x_i), \qquad j = 1,..., p.$$

Thus by (5.2), S solves Problem 1.1. ▮

6. Examples

In this section we give several examples to illustrate the simplicity of the method. We consider solving the (0, 2) interpolation problem with $\Delta = \{(i-1)/n\}_{i=1}^{n+1}$ using piecewise polynomials of degree 4. In this case $I = (0, 2)$. There are three possibilities for the choice of the set K of derivatives to be estimated by a quadrature rule—we may take K to be $\{1\}$, $\{3\}$, or $\{4\}$. This leads to three different algorithms which we discuss in Examples 6.1–6.3 below.

Example 6.1 ($K = \{4\}$). To describe the algorithm it suffices to give the quadrature formulae for estimating the values $\{s_i^4\}_1^{n+1}$. Let

$$\begin{aligned} s_i^4 &= (f_1 - 4f_2 + 6f_3 - 4f_4 + f_5)/h^4, && i = 1, 2 \\ s_i^4 &= (f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2})/h^4, && i = 3,..., n-1 \\ s_i^4 &= (f_{n-3} - 4f_{n-2} + 6f_{n-1} - 4f_n + f_{n+1})/h^4, && i = n. \end{aligned} \tag{6.1}$$

Discussion. In choosing these quadrature formulae, we have chosen l_i as in (4.7) and determined the coefficients by solving the systems (4.6). The method gives up to order $O(h^4)$ accuracy. This accuracy is achieved when $f \in L_\infty^4[0, 1]$. ▮

Example 6.2 ($K = \{3\}$). Let

$$\begin{aligned} s_1^3 &= (-3f_1^2 + 4f_2^2 - f_3^2)/2h \\ s_i^3 &= (f_{i+1}^2 - f_{i-1}^2)/2h, \qquad i = 2,..., n \end{aligned} \tag{6.2}$$

Discussion. In this example we have elected to estimate the third derivatives by using the data on the second derivatives, rather than the function values $f_1^0, ..., f_{n+1}^0$ themselves. By using the second derivatives we have simpler quadrature formulae involving only three data points instead of five. Even more local formulae could be constructed by using both the f_i^0 and f_i^2's. This method gives accuracy up to order $O(h^4)$. ∎

EXAMPLE 6.3. ($K = \{1\}$). Let

$$
\begin{aligned}
s_1^1 &= (-25f_1 + 48f_2 - 36f_3 + 16f_4 - 3f_5)/12h \\
s_2^1 &= (-3f_1 - 10f_2 + 18f_3 - 6f_4 + f_5)/12h \\
s_i^1 &= (f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2})/12h, \qquad i = 3, ..., n-1 \\
s_n^1 &= (f_{n-3} - 6f_{n-2} + 18f_{n-1} - 10f_n - 3f_{n+1})/12h \qquad (6.3)
\end{aligned}
$$

Discussion. As in Examples 6.1 and 6.2, this method gives up to order accuracy $O(h^4)$. In contrast with Example 6.2, here we must use the function values as we cannot estimate first derivatives in terms of second derivatives. ∎

7. REMARKS

1. The methods proposed here produce piecewise polynomials of order m with global smoothness $C^{\nu_1}[x_1, x_{n+1}]$. In addition, by the construction the derivatives from left and right of orders $\nu_2, ..., \nu_p$ have been forced to match at each x_i, $i = 2, ..., n$. The solutions are examples of g-splines (cf. [12]).

2. If it is desired to solve the lacunary interpolation problem with a function s which has more global smoothness, we can proceed as follows. Suppose we want $s \in C^\rho[x_1, x_{n+1}]$ with $\rho > \nu_1$. Then given $I = \{\nu_1 < \cdots < \nu_p\}$, let $\tilde{I} = \{\tilde{\nu}_1 < \cdots < \tilde{\nu}_{\tilde{p}}\}$ be chosen such that $\tilde{I}$ contains $\nu_1, \nu_1 + 1, ..., \rho$. Now if we solve the lacunary interpolation problem corresponding to $\tilde{I}$ where the missing data is supplied by using quadrature formulae, then the resulting g-spline solution $\tilde{s}$ will belong to $C^\rho[x_1, x_{n+1}]$ and will also solve the original lacunary interpolation problem.

3. The methods proposed here are completely local as compared with polynomial and spline methods where all parameters must be determined at once by solving one large linear system of equations. Another advantage of our g-spline methods as compared to the usual spline methods (cf. [2, 8–10, 14, 15]) is that here there is no need to specify some kind of end conditions.

4. In Theorems 3.2 and 3.4 we have given error bounds which hold on the entire interval $[x_1, x_{n+1}]$. Since our methods are local, it is also possible to give *local* error bounds valid on the individual intervals $[x_i, x_{i+1}]$, $i = 1, ..., n$. Typically, to get an error bound on $[x_i, x_{i+1}]$ we would assume f is smooth on a slightly larger interval $[x_{l_i}, x_{r_i}]$ which includes all the points involved in the quadrature formulae used at x_i.

5. Using piecewise polynomials of order m, it is well known that there is a saturation phenomenon (cf. [12]) which prevents getting better than $O(h^m)$ approximation power (except for a small saturation class), no matter how smooth the function f being approximated may be. The methods constructed here achieve this optimal approximation power.

6. The present paper extends the work of Fawzy [3–7] in several ways. Here we give a general construction which works for all lacunary interpolation problems and all orders m, we allow arbitrarily spaced $x_1, ..., x_{n+1}$, and we give error bounds for all smoothness classes $C^{\sigma-1}$ and L_p^σ instead of just C^m.

7. Using Taylor expansion methods (cf. [3–7]) it is possible to get explicit constants in the error bounds for explicit methods.

8. The techniques introduced here could also be used to solve more general lacunary interpolation problems where the data being specified vary from point to point; i.e., we would choose $I_i = \{v_1^i < \cdots < v_{p_i}^i\}$ for $i = 1, ..., n+1$.

9. In Section 4 we have examined quadrature formulae of the form $Qf = \sum_{j=1}^{n+1} \alpha_j f(x_j)$. It would also be possible to design quadrature formulae using a mixture of values of f and its derivatives $f^{(v_2 - v_1)}, ..., f^{(v_p - v_1)}$. Although the design of such formulae is more complicated, they would have the advantage of being more local; i.e., our estimate of $f^{(v)}(x_i)$ would require data at points closer to x_i.

10. The method described here is easily programmed. We have tested a FORTRAN package extensively at Texas A & M. For some numerical results on a typical test problem, see [7].

11. It is not hard to show that the lacunary spline methods discussed here are stable in the sense that if the data are perturbed by a small amount, then the spline also varies only by a small amount (cf. [14]) for precise statements of such stability assertions.

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